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Embedding of Posets into Lattices in Quantum Logic[1]

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It has been proposed that some posets of quantum logic could be embedded into lattices in order to recover the lattice structure avoiding the introduction of "ad hoc" axioms. We consider here the embedding ϕ_S of any poset S into the complete lattice $\mathscr{L}_S$ of its closed ideals ("normal" embedding of S) and show that ϕ_S can be characterized (up to a lattice isomorphism) either by means of a "density" property or by means of a "minimality" property. Both of these suggest that the normal embedding satisfies some intuitive conditions which make it preferable with respect to other possible embeddings of S. We consider the poset $\mathscr{E}$ of all the "effects" associated to yes-no experiments and briefly comment on the application of the normal embedding in this case. The possibility of giving a physical interpretation to the elements of $\mathscr{L}_{\mathscr{E}}$ is also discussed.

1. INTRODUCTION

It has been suggested by some authors that the methods for embedding posets into lattices can find some use in quantum logic (Q.L.) as a way of bypassing a common difficulty of many axiomatic approaches, and precisely that of finding physical arguments which can support the assumption that the basic poset, which can be different in the various approaches, is a lattice (e.g., Beltrametti and Cassinelli, 1976; Jauch, 1968; Mackey, 1963; and Piron, 1976). In particular, the well-known "normal" embedding ϕ_S of a poset S into the complete lattice $\mathscr{L}_S$ of all its closed ideals (sometimes referred to in the literature as the "completion by cuts" of S) that makes every $a \in S$ correspond to the principal ideal generated by a itself has been explicitly proposed by K. Bugaiska and S. Bugaiski (1973) as the "natural" one for the "propositional" logic of any physical system; the same authors have proved that this embedding preserves some structure properties of S that are important in Q.L. Similarly, W. Guz has proposed to embed suitably

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any logic into its "phase geometry" and has shown that this embedding coincides (up to a lattice isomorphism) with the normal embedding whenever some reasonable assumptions are introduced on the set of the pure states of the physical system.

However, there are infinitely many ways of embedding a poset into a lattice, so that the choice of a particular embedding as physically meaningful ought to rest, in our opinion, upon deeper arguments; in addition we think it desirable that these arguments do not depend upon properties that hold only in the framework of some particular approaches to quantum theory.

In the present work we consider any embedding ϕ of a poset S into a complete lattice $\mathscr{L}$. Our first result (see Section 2, Proposition 1) shows that the following conditions are equivalent.

(i) ϕ coincides, up to a lattice isomorphism, with the normal embedding ϕ_S of S.

(ii) Density property: Every element of $\mathscr{L}$ is the lower upper bound (l.u.b.) of some subset of $\phi(S)$ and the greatest lower bound (g.l.b.) of some other subset of $\phi(S)$.

(iii) Minimality property: Any embedding ψ of S into a complete lattice $\mathscr{M}$ is such that $\psi = h \circ \phi$, with $h: \mathscr{L} \to \mathscr{M}$ an order isomorphism of $\mathscr{L}$ onto $h(\mathscr{L})$.

These equivalent properties have an important intuitive interpretation whenever S is the logic of some physical system. Indeed the equivalence between (i) and (ii) shows that $\mathscr{L}_S$ is the unique (up to a lattice isomorphism) complete lattice extension of S which guarantees that for any object $A \in \mathscr{L}_S \backslash S$ two classes of elements of S (i.e., of "physical objects") exist such that A is the unique element of $\mathscr{L}_S$ which separates them. The equivalence between (i) and (iii) guarantees that $\mathscr{L}_S$ is obtained from S by adjunction of the "minimum possible number" of elements.

The above interpretations are arguments in favor of the normal embedding which do not depend, as desired, upon the further assumptions on the basic poset S that are introduced in the various axiomatic approaches to Q.L.

In addition, we remark that the usual requirement in Q.L. that the lattice be complete is a mathematical requisite without direct physical justification and is imposed in order to obtain more manageable structures; thus one can believe that an approach is more realistic which avoids introducing it. At the end of Section 2 we show that every embedding of S into a lattice $\mathscr{L}$ (not necessarily complete) coincides, up to a lattice isomorphism, with the embedding of S in some suitable sublattice of $\mathscr{L}_s$ iff S is dense in $\mathscr{L}$. This seems again an argument in favor of the normal embedding.

In Section 3 we remark that the above results suggest, in particular, how to recover an intuitively legitimated complete lattice structure starting from the poset $\mathscr{E}$ of the "effects" (these are equivalence classes of the yes–no experiments that can be performed on a given physical system) or from any subset of $\mathscr{E}$ which may be important in Q.L., and briefly sketch the lines that can be followed in this case. We close our work with some comments about the physical interpretation of the elements of the lattice $\mathscr{L}_{\mathscr{E}}$ of the closed ideals of $\mathscr{E}$; in particular, we show that those elements which do not correspond to effects could perhaps be interpreted as equivalence classes of yes–no devices for which a probability of the yes outcome can be defined only if it is 0 or 1.

2. NORMAL EMBEDDING OF POSETS INTO LATTICES

As we have anticipated in the Introduction, we will firstly deal in this section with the problem of characterizing the embedding of every poset into the complete lattice of its closed ideals by means of equivalent conditions. The equivalence between (i) and (ii) essentially rephrases Theorem 9 in Chap. 3 of Skorniakov's book (Skorniakov, 1977). In the present work an alternative proof follows immediately from some preliminary lemmas and from our proof of the equivalence between (i) and (iii). This last equivalence, in turn, could be deduced by Skorniakov's Theorem 8 (Skorniakov, 1977); we give here a direct proof of it.

Definition 1. Let $\mathscr{L}$ be any lattice, S any subposet of $\mathscr{L}$. Then, we say that S is a dense subposet of $\mathscr{L}$ whenever any $a \in \mathscr{L}$ is the greatest lower bound (briefly, g.l.b.) of some subset of S and the lower upper bound (briefly, l.u.b.) of some other subset of S.

Definition 2. Let S be any poset. For any $A \subseteq S$ we put

$$A^{\triangle} = \{a \in S | a \leq b \ \forall b \in A\}$$

$$A^{\triangledown} = \{a \in S | b \leq a \ \forall b \in A\}$$

We recall now some definitions and results which are standard in poset and lattice theory (e.g., Birkhoff, 1967).

Let S be a poset, $A \subseteq S$. Then, A is usually said to be a closed ideal of S if $(A^{\triangledown})^{\triangle} = A$, a closed filter if $(A^{\triangle})^{\triangledown} = A$. Let $\mathscr{L}_S$ and $\tilde{\mathscr{L}}_S$ be, respectively, the sets of all the closed ideals and of all the closed filters of S, partially ordered by set theoretical inclusion. Then, $\mathscr{L}_S$ and $\tilde{\mathscr{L}}_S$ are complete lattices, and the mapping

$$\xi: A \in \mathscr{L}_S \to A^{\triangledown} \in \tilde{\mathscr{L}}_S$$

is a dual isomorphism.

Furthermore, the injective mapping

$$\phi_S: a \in S \to \{a\}^{\triangle} \in \mathscr{L}_S$$

is an order isomorphism of S onto $\phi_S(S)$, preserves l.u.b. and g.l.b. existing in S and will be called the "normal" embedding of S; it is apparent that $\phi_S(S)$ coincides with the poset of all the principal ideals of S.

Then, we can state the following lemmas:

Lemma 1. Let S be any poset, let ϕ_S be the normal embedding of S and let us make reference to Definition 1. Then, $\phi_S(S)$ is dense in the lattice $\mathscr{L}_S$ of the closed ideals of S.

Proof. Let A be a closed ideal of S and let $\cup$ and $\cap$, respectively, denote set-union and set-intersection. Then, by making reference to Definition 2, we easily get $A = \bigcup_{a\in A}\{a\}^{\triangle} = \bigcap_{a\in A^{\triangledown}}\{a\}^{\triangle}$, so that A is l.u.b. in $\mathscr{L}_S$ of the set $\{\{a\}^{\triangle}|a\in A\}$ and g.l.b. of the set $\{\{a\}^{\triangle}|a\in A^{\triangledown}\}$, both consisting of principal ideals of S. Since $\phi_S(S)$ coincides with the poset of all the principal ideals of S, our statement is proved. ■

Lemma 2. Let $\mathscr{L}$ be a (nonnecessarily complete) lattice, let us denote by $\vee$ and $\wedge$, respectively, join and meet in $\mathscr{L}$ and let us make reference to Definition 1. Let S be a dense subposet of $\mathscr{L}$. Then, for every $y\in\mathscr{L}$

$$y = \bigwedge_{\substack{x\in S\\ x\geq y}} x = \bigvee_{\substack{x\in S\\ x\leq y}} x$$

Moreover, the set $A_y = \{x\in S|x\leq y\}\subseteq S$ is a closed ideal of S and the set $\{x\in S|x\geq y\}\subseteq S$ coincides with the closed filter $A_y^{\triangledown}$ of S.

Proof. The first statement of the lemma directly follows from the definition of density.

The second statement follows from the first statement and from the implications

$$x\in A_y^{\triangledown}\Leftrightarrow x\in S \text{ and } x\geq z\ \forall z\in A_y \Leftrightarrow x\in S \text{ and } x\geq y$$

$$x\in (A_y^{\triangledown})^{\triangle}\Leftrightarrow x\in S \text{ and } x\leq z\ \forall z\in A_y^{\triangledown}\Leftrightarrow x\in A_y \qquad ■$$

Lemma 3. Let S be any poset, ψ any embedding of S into a complete lattice $\mathcal{M}$, $\mathscr{L}_S$ the complete lattice of the closed ideals of S, ϕ_S the normal embedding of S. Let us denote by $\vee$ the join in $\mathcal{M}$, and let us make reference to Definitions 1 and 2. Then, the mapping

$$f\colon A\in\mathscr{L}_S\to f(A) = \bigvee_{x\in\psi(A)} x\in\mathcal{M}$$

is an order isomorphism of $\mathscr{L}_S$ onto $f(\mathscr{L}_S)\subseteq\mathcal{M}$ such that

$$\psi = f\circ\phi_S$$

Moreover, f reduces to a lattice isomorphism of $\mathscr{L}_S$ onto $\mathscr{M}$ whenever $\psi(S)$ is dense in $\mathscr{M}$.

Proof. The mapping f is trivially monotonic. Let us prove first that it is an order isomorphism of $\mathscr{L}_S$ onto $f(\mathscr{L}_S)$.

Let $A \subseteq S$ be a closed ideal of S. Since the mapping ψ is an embedding of S into $\mathscr{M}$, we get

$$\psi(A^\nabla) = \{y \in \psi(S) | y \geq f(A)\}$$

hence, recalling the definition of $f(A)$,

$$\begin{aligned}\psi(A) = \psi((A^\nabla)^\triangle) &= \{x \in \psi(S) | x \leq y \text{ for any } y \in \psi(A^\nabla)\}\\ &= \{x \in \psi(S) | x \leq f(A)\}\end{aligned}$$

Thus, for every $A, B \in \mathscr{L}_S$,

$$f(A) = f(B) \Rightarrow \psi(A) = \psi(B) \Rightarrow A = B$$

hence f is injective. Furthermore, let $x, y \in f(\mathscr{L}_S)$ so that $x = f(A)$ and $y = f(B)$ for some $A, B \in \mathscr{L}_S$, and let $x \leq y$. Then, we get

$$x \leq y \Rightarrow f(A) \leq f(B) \Rightarrow \psi(A) \subseteq \psi(B) \Rightarrow A \subseteq B \Rightarrow f^{-1}(x) \leq f^{-1}(y)$$

Therefore, our first statement is proved.

Let us show now that $\psi = f \circ \phi_S$. To this end, let us consider any $a \in S$. Then, we have

$$f(\phi_S(a)) = f(\{a\}^\triangle) = \bigvee_{x \in \psi(\{a\}^\triangle)} x = \bigvee_{\substack{x \in \psi(S)\\ x \leq \psi(a)}} x = \psi(a)$$

Finally, let us show that f is a lattice isomorphism of $\mathscr{L}_S$ onto $\mathscr{M}$ whenever $\psi(S)$ is dense in $\mathscr{M}$. Indeed, in this case, for every $y \in \mathscr{M}$ the set $A_y = \{x \in \psi(S) | x \leq y\}$ is a closed ideal of $\psi(S)$ because of Lemma 2, hence $\psi^{-1}(A_y)$ is a closed ideal of S. Then, $f(\mathscr{L}_S) = \mathscr{M}$. Thus, f is an order isomorphism of $\mathscr{L}_S$ onto $\mathscr{M}$, and our statement is proved. ■

Remark. In the statement of Lemma 3 the mapping f could just as well be substituted everywhere by the mapping

$$f' : A \in \mathscr{L}_S \to f'(A) = \bigwedge_{x \in \psi(A)} x \in \mathscr{M}$$

as can easily be seen by considering the dual lattice $\tilde{\mathscr{L}}_S$ of $\mathscr{L}_S$.

By comparing the definitions of f and f' we obtain at once

$$f(A) \leq f'(A) \; \forall A \in \mathscr{L}_S$$

Of course, $f = f'$ whenever $\phi(S)$ is dense in $\mathscr{M}$.

We can now state the following Proposition.

Proposition 1. Let S be any poset, ϕ any embedding of S into a complete lattice $\mathscr{L}$, ϕ_S the normal embedding of S. Let us make reference to Definitions 1 and 2. Then, the following conditions are equivalent:

(i) $\phi = g \circ \phi_S$ with g a lattice isomorphism.

(ii) $\phi(S)$ is dense in $\mathscr{L}$.

(iii) For any embedding ψ of S into a complete lattice $\mathscr{M}$, an order isomorphism $h: \mathscr{L} \to h(\mathscr{L}) \subseteq \mathscr{M}$ exists such that $\psi = h \circ \phi$.

Proof. Condition (i) implies condition (iii) because of the first statement in Lemma 3.

Condition (ii) implies condition (i) because of the second statement in Lemma 3.

It remains to show that condition (iii) implies condition (ii). To this end, let us assume that (iii) holds and let us put $\psi = \phi_S$. Then, $h(\mathscr{L})$ is a subposet of $\mathscr{L}_S$, hence $h(\phi(S)) = \phi_S(S)$ is dense in $h(\mathscr{L})$ because of Lemma 1, so that $\phi(S) = h^{-1}(\phi_S(S))$ must be dense in $\mathscr{L}$. ∎

As we have anticipated in the Introduction, our next statement connects the normal embedding of any poset S with any dense embedding of S into a (nonnecessarily complete) lattice.

Proposition 2. Let S be any poset, ϕ any embedding of S into a lattice $\mathscr{L}$, $\mathscr{L}_S$ the complete lattice of the closed ideals of S, ϕ_S the normal embedding of S, and let us make reference to Definition 1. Then, the following statements are equivalent:

(i) a sublattice $\mathscr{L}_1$ of $\mathscr{L}_S$ exists such that $\phi = g \circ \phi_S$, with g a lattice isomorphism of $\mathscr{L}_1$ onto $\mathscr{L}$.

(ii) $\phi(S)$ is dense in $\mathscr{L}$.

Proof. The implication (i)$\Rightarrow$(ii) is an immediate consequence of Lemma 1. Therefore, we will limit ourselves to prove the implication (ii)$\Rightarrow$(i).

Let us denote join and meet, both in $\mathscr{L}$ and in $\mathscr{L}_S$, with $\vee$ and $\wedge$, respectively (we recall that $\cup$ means set union in this paper); furthermore, for any $y \in \mathscr{L}$, we put $A_y = \{x \in \phi(S) | x \leq y\}$. Let $\phi(S)$ be dense in $\mathscr{L}$. Then, we obtain from Lemma 2 that A_y is a closed ideal of $\phi(S)$ and that $y = \bigvee_{x \in A_y} x$; hence, it follows at once that $\phi^{-1}(A_y) \in \mathscr{L}_S$ and that the mapping

$$l: y \in \mathscr{L} \to \phi^{-1}(A_y) \in \mathscr{L}_S$$

is a poset isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1 = l(\mathscr{L}) \subseteq \mathscr{L}_S$. Let us show that $\mathscr{L}_1$ is a sublattice of $\mathscr{L}_S$.

Let $Y, Z \in \mathscr{L}_1$; then, two elements $y, z \in \mathscr{L}$ exist such that $Y = \phi^{-1}(A_y)$, $Z = \phi^{-1}(A_z)$. By making use of the first statement of Lemma 2, we get

$w = y \vee z = \bigvee_{x \in A_y \cup A_z} x$. Let us consider the closed ideal $W = \phi^{-1}(A_w) \in l(\mathscr{L})$. We have trivially $W \supseteq Y$, $W \supseteq Z$, hence $W \supseteq Y \vee Z$. Furthermore, for every $a \in S$,

$$a \in W \Rightarrow a \leq b \ \forall b \in (Y \cup Z)^{\nabla}$$
$$\Rightarrow a \leq b \ \forall b \in (Y \vee Z)^{\nabla} \Rightarrow a \in ((Y \vee Z)^{\nabla})^{\Delta} = X \vee Z$$

hence $W \subseteq Y \vee Z$. Thus, $Y \vee Z = W$, i.e., the join $Y \vee Z$ belongs to $\mathscr{L}_1$. Similarly, we can prove that $Y \wedge Z \in \mathscr{L}_1$. Then, $\mathscr{L}_1$ is a sublattice of $\mathscr{L}_S$, as stated.

By setting $g = l^{-1}$, our statement (i) follows at once ■

We would like to close this section recalling the following results obtained by Bugaiska and Bugaiski (1973).

First, whenever S is an orthoposet, the orthocomplementation can be extended to the complete lattice $\mathscr{L}_S$ of its closed ideals. Second, whenever S is an atomic poset, then also $\mathscr{L}_S$ is atomic. Finally, whenever S is an orthomodular poset which satisfies some further axioms (like the "projection postulate") which derive from a physical interpretation of S, then $\mathscr{L}_S$ also is orthomodular.

3. NORMAL EMBEDDING IN QUANTUM LOGIC

We will firstly comment in this section on the application of the normal embedding in Q.L. whenever the poset of the effects is considered.

To this end we recall some preliminary definitions [for further details, see Garola (1980) and Garola and Solombrino (1983)].

We consider the concepts of physical system, device with dichotomic outcome, state of the system, and probability as primitive.

We call "yes-no experiment" any device e with dichotomic outcome such that a probability $\alpha(e)$ can be attributed to the yes outcome whenever e is performed on the system in the state α, and denote by e' the experiment obtained from e by reversing the roles of the yes and no outcomes, by E the set of all the yes-no experiments, by $\mathscr{S}$ the set of all the states for a given system.

We introduce in E a preorder relation $\leq$, by setting

$$\text{for any } e, f \in E,\ e \leq f \Leftrightarrow \alpha(e) \leq \alpha(f) \text{ for every } \alpha \in \mathscr{S}$$

and the equivalence relation $\sim$ induced by $\leq$. Furthermore, we assume that a "certainly true" experiment e_I and a "certainly false" experiment e_0 exist in E, respectively, characterized by $\alpha(e_I) = 1$ and $\alpha(e_0) = 0$ for every $\alpha \in \mathscr{S}$.

We call "set of the effects" the set $\mathscr{E} = E/\sim$, denote again by $\leq$ the order in $\mathscr{E}$ induced by the preorder in E, and for every $e \in E$, $a = [e]_{\sim} \in \mathscr{E}$ we put $\alpha(a) = \alpha(e)$.

Let us consider now $\mathscr{E}$ with the order $\leq$. In this poset, g.l.b. and l.u.b. do not necessarily exist for every pair of elements. We can introduce the normal embedding $\phi_{\mathscr{E}}$ of $\mathscr{E}$ into the lattice $\mathscr{L}_{\mathscr{E}}$ in order to recover a complete lattice structure; then, our arguments in the Introduction assure that this embedding satisfies some simple intuitive conditions.

Nevertheless, the lattice $\mathscr{L}_{\mathscr{E}}$ need not be the most suitable structure for the foundations of Q.L.; indeed, there are properties of the "logic" of a physical system which are important in most axiomatic approaches and that are neither shared by $\mathscr{E}$ nor by $\mathscr{L}_{\mathscr{E}}$. Since every property Π in the power set $\mathscr{P}(\mathscr{E})$ of $\mathscr{E}$ characterizes a class of subsets of $\mathscr{E}$, we could restrict ourselves to one, say, $\mathscr{G}$, of these subsets (if not trivial) in order to obtain a richer structure. However, $\mathscr{G}$, like $\mathscr{E}$, need not be a lattice; thus, in order to avoid the introduction of assumptions that could be hard to justify from a physical point of view, we can again recover a lattice structure by making use of the normal embedding of $\mathscr{G}$ in $\mathscr{L}_{\mathscr{G}}$. Of course, an additional requirement must be introduced now, i.e., that the property Π extends to $\mathscr{G}$.

As a first example, let us consider the property of being orthocomplemented by the involutory antiautomorphism $\eta:[e]_{\sim}\to[e']_{\sim}$ of $\mathscr{E}$ (we recall that in any logic the existence of an orthocomplementation is fundamental because it is usually interpreted as the "negation" in that logic). We have proved (Garola, 1980) that the class of the subposets of $\mathscr{E}$ which share this property, ordered by set theoretical inclusion, has a maximum, say, $\mathscr{E}_0$ (we recall that $\mathscr{E}_0$ is the quotient set $E_0/\sim$, with E_0 the subset of E whose elements are e_O, e_I and every yes-no experiment $e\in E$ satisfying the condition "a pair (α, β) of states exist such that $\alpha(e)<1/2$ and $\beta(e)>1/2$") but for nonclassical system one cannot generally assume that $\mathscr{E}_0$ is a lattice. Nevertheless, one can obtain a complete lattice by introducing the normal embedding $\phi_{\mathscr{E}_0}$ of $\mathscr{E}_0$ into the lattice $\mathscr{L}_{\mathscr{E}_0}$. Then, the orthocomplementation can be extended to $\mathscr{L}_{\mathscr{E}_0}$ because of the results reported at the end of Section 2, the orthocomplementation in $\mathscr{L}_{\mathscr{E}_0}$ being such that its restriction to $\phi_{\mathscr{E}_0}(\mathscr{E}_0)$ coincides with $\phi_{\mathscr{E}_0}\circ\eta\circ\phi_{\mathscr{E}_0}^{-1}$.

As a further trivial example, we quote the poset $\mathscr{F}=F/\sim\subseteq\mathscr{E}_0$, with F the set of all the ideal, first-kind measurements (e.g., Jauch, 1968). This set is commonly accepted to be a complete, orthomodular, atomic lattice satisfying the covering law. Then, the normal embedding of $\mathscr{F}$ obviously coincides with $\mathscr{F}$ itself and preserves all its properties.

We observe that the posets considered in this section can be arranged so as to obtain the descending chain

$$\mathscr{E}\supseteq\mathscr{E}_0\supseteq\cdots\supseteq\mathscr{F}$$

Here, the empty spaces indicate the possibility of inserting in the chain further subposets of $\mathscr{E}$ which share only some of the properties of $\mathscr{F}$. Then, the lattices of the closed ideals of these posets form the descending chain

$$\mathcal{L}_{\mathscr{E}} \supseteq \mathcal{L}_{\mathscr{E}_0} \supseteq \cdots \supseteq \mathcal{L}_{\mathscr{F}} = \mathscr{F}$$

(we notice that the above inclusions between lattices do not necessarily preserve l.u.b. and g.l.b., i.e., join and meet). Should an orthoposet $\mathscr{G}$ be inserted between $\mathscr{E}_0$ and $\mathscr{F}$ in the first chain above, endowed with such properties like atomicity, or orthomodularity, or both, then the lattice $\mathcal{L}_{\mathscr{G}}$ would take the corresponding place in the second chain and preserve, because of the results and under the assumptions reported in Section 2, the properties of $\mathscr{G}$.

We would like to close our paper with some remarks about the physical interpretation of the elements of $\mathcal{L}_{\mathscr{E}}$. It will be useful to inquire first into the possibility of attributing a physically motivated probability to every $A \in \mathcal{L}_{\mathscr{E}}$ for every state of the system.

For every $\alpha \in \mathscr{S}$, $a \in \mathscr{E}$, $A = \phi_{\mathscr{E}}(a) \in \phi_{\mathscr{E}}(\mathscr{E}) \subseteq \mathcal{L}_{\mathscr{E}}$, the probability $\alpha(A)$ of A is naturally defined by the equation

$$\alpha(A) = \alpha(a)$$

Let us consider any $A \in \mathcal{L}_{\mathscr{E}}$. We recall from Lemma 1 of Section 2 that

$$A = \text{l.u.b.}\, \{B \in \phi_{\mathscr{E}}(\mathscr{E}) | B \leq A\} = \text{g.l.b.}\, \{B \in \phi_{\mathscr{E}}(\mathscr{E}) | A \leq B\}$$

Then, for every $\alpha \in \mathscr{S}$, we put

$$\alpha_{\vee}(A) = \text{l.u.b.}\, \{\alpha(B) | B \in \phi_{\mathscr{E}}(\mathscr{E}),\, B \leq A\}$$

$$\alpha_{\wedge}(A) = \text{g.l.b.}\, \{\alpha(B) | B \in \phi_{\mathscr{E}}(\mathscr{E}),\, A \leq B\}$$

Of course, whenever $A \in \phi_{\mathscr{E}}(\mathscr{E})$, $\alpha_{\vee}(A) = \alpha_{\wedge}(A) = \alpha(A)$. More generally, for any $A \in \mathcal{L}_{\mathscr{E}}$ it is $\alpha_{\vee}(A) \leq \alpha_{\wedge}(A)$, but we cannot say that $\alpha_{\vee}(A) = \alpha_{\wedge}(A)$ for every $\alpha \in \mathscr{S}$.

Thus, we see that we have in the general case no mathematical support for attributing a uniquely defined "probability" $\alpha(A)$ to A for every state α. Yet, this does not prohibit that for any A some subclass $\mathscr{S}_A \subseteq \mathscr{S}$ of states exists such that $\alpha(A)$ can be defined for every $\alpha \in \mathscr{S}_A$. Indeed, for every $A \in \mathcal{L}_{\mathscr{E}}$ let us consider the "certainly yes" domain $\mathscr{S}_1(A)$ and the "certainly not" domain $\mathscr{S}_0(A)$, respectively, defined by the equations

$$\mathscr{S}_1(A) = \{\alpha \in \mathscr{S} | \alpha_{\vee}(A) = 1\}$$

$$\mathscr{S}_0(A) = \{\alpha \in \mathscr{S} | \alpha_{\wedge}(A) = 0\}$$

Then, for every $\alpha \in \mathscr{S}_1(A)$, $\alpha_{\vee}(A) = \alpha_{\wedge}(A) = 1$, so that we can attribute to A the probability $\alpha(A) = 1$. Analogously, for every $\alpha \in \mathscr{S}_0(A)$, we can attribute to A the probability $\alpha(A) = 0$.

Let us come now to the physical interpretation of the elements of $\mathcal{L}_{\mathscr{E}}$. Trivially, any $A \in \phi_{\mathscr{E}}(\mathscr{E})$ can be interpreted as an effect, i.e., an equivalence class of elements of $\mathscr{E}$. Of course, this is no more possible for an element $A \in \mathcal{L}_{\mathscr{E}} \backslash \phi_{\mathscr{E}}(\mathscr{E})$. Yet, our arguments above about probability show that every such A could perhaps be associated to some physical dichotomic device $q \in E$ which attribute a probability $\alpha(q)$ to the yes outcome only if the state α either belongs to the "certainly yes" domain $\mathcal{S}_1(A)$ [then, $\alpha(q)=1$] or to the "certainly not" domain $\mathcal{S}_0(A)$ [then, $\alpha(q)=0$], but no probability whenever α does not belong to $\mathcal{S}_1(A)$ or to $\mathcal{S}_0(A)$. This interpretation of the elements of $\mathcal{L}_{\mathscr{E}} \backslash \phi_{\mathscr{E}}(\mathscr{E})$ seems, at first glance, rather odd. However, we observe that devices with the property considered above are actually introduced in some approach to quantum theory (Piron, 1976; Aerts, 1980-81).

In fact, in Piron's treatment no reference is made to the concept of probability when introducing the fundamental concept of question (dichotomic device); rather, any question characterizes a set of states, say, $\mathcal{S}_1(q)$, such that q is "certainly true" whenever the system is in a state $\alpha \in \mathcal{S}_1(q)$, and a set of states, say $\mathcal{S}_0(q)$, such that q is "certainly false" whenever the system is in a state $\alpha \in \mathcal{S}_0(q)$. Then, physical devices such that a probability is defined only if it is 0 or 1 may be considered questions [although, by assuming some further postulates, a probability can be attributed to every "proposition" in every state of the system (Piron, 1976)]. An example of such a device is the question $q = \Pi_i q_i$ introduced in Piron's book (with $\{q_i\}$ a family of questions), because of the prescription of testing an "arbitrary one" of the q_i in order to test q (we recall that this question plays an important role in Piron's approach, since it is used by the author in order to construct the meet of any family of propositions, so as to prove that the poset of the propositions is a complete lattice).

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